A DISCUSSION OF THE EQUATION OF THE SECOND DEGREE IN TWO VARIABLES.*

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1. The equation of a conic section is by Boscovich's definition

$$(x - x')^2 + (y - y')^2 = e^2 (x\lambda + y\mu - p)^2, \qquad (A)$$

where (x', y') is the focus, e is a constant, and $x\lambda + y\mu - p = 0$ is the normal equation of the directrix. Multiplying through by ρ and assuming

$$\rho(1 - e^2\lambda^2) = a, \qquad (1)$$
$$-\rho e^2\lambda\mu = h, \qquad (2)$$
$$\rho(1 - e^2\mu^2) = b, \qquad (3)$$
$$\rho(e^2 p\lambda - x') = g, \qquad (4)$$
$$\rho(e^2 p\mu - y') = f, \qquad (5)$$
$$\rho(x'^2 + y'^2 - e^2p^2) = c, \qquad (6)$$

equation (A) becomes

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad (B)$$

The co-efficients a, b, c, f, g, h are not conditioned by these equations (1) to (6), as there are six unknowns in the equations. Solving for these unknowns, we get

$$\left.\begin{aligned}
e^2 &= \frac{2R}{R+s}, \\
\mu &= +\sqrt{\frac{R+d}{2R}}, \\
\lambda &= \theta\sqrt{\frac{R-d}{2R}}, \\
\lambda\mu &= -\frac{h}{R}, \\
p &= \frac{f^2 + g^2 - \frac{1}{2}c(R+s)}{R(f\mu + g\lambda) \mp \sqrt{-R\Delta}} \\
&= \frac{2f\mu + 2g\lambda}{R-s} \pm \sqrt{\frac{-4\Delta}{R(R-s)^2}} \\
&= p' \pm p'', \text{ say,}
\end{aligned}\right\} \qquad (7)$$

*An abstract of the following paper was presented to the University Mathematical Society of Baltimore in March, 1883, and it has lately been suggested to me that it might be worth while to publish it more in detail for the benefit of a more general class of readers. The method of discussion, though quite far removed from what may be designated as the modern methods, is perhaps the most elementary that can be given and for some purposes is by far the most convenient.

$$x' = e^2\left\{p\lambda - \frac{g}{R}\right\}$$
$$= \frac{bg - fh}{h^2 - ab} \pm \frac{\sqrt{-\frac{1}{2}\Delta(R-d)}}{h^2 - ab}, \qquad (7\ cont.)$$
$$y' = e^2\left\{p\mu - \frac{f}{R}\right\}$$
$$= \frac{af - gh}{h^2 - ab} \pm \frac{\sqrt{-\frac{1}{2}\Delta(R+d)}}{h^2 - ab};$$

where for brevity we write

$$s = a + b,$$
$$d = a - b,$$
$$R = \pm\sqrt{d^2 + 4h^2} = \pm\sqrt{s^2 + 4(h^2 - ab)},$$
$$\theta = \text{sign of } -\frac{h}{R},$$
$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2.$$

We thus see that for each of the two (real) values of R there is one real value of e^2; one real value of λ and of μ (and consequently one real value of the angle α); two values of p, one pair of values being real, the other pair being imaginary, unless (1) $\Delta = 0$, in which case the members of each pair are real and equal, or (2) $R = 0$, when all four values of p become infinite; and one value of x', y' corresponding to each value of p, real or imaginary according as p is real or imaginary. The two values of the angle α differ by 90°. For let λ_1, μ_1 be the values of λ, μ obtained by substituting the positive value of R and λ_2, μ_2 be the other values, and let θ' be the sign of $-h$. Then

$$\lambda_1 = \theta'\mu_2 \text{ and } \lambda_2 = -\theta'\mu_1,$$

whence α_1 and α_2 differ by 90°.

We have thus seen that the general equation of the second degree (B) is the equation of a conic section, and have found its foci, directrices, and constant ratio e in terms of its co-efficients. The results show that a conic has four directrices, two entirely real and parallel to each other, the other two perpendicular to the first pair at an imaginary distance from each other, unless (1) $\Delta = 0$, in which case the members of each pair are real and coincident, or (2) $R = 0$, when the directrices are all real and at an infinite distance from the origin. To each directrix corresponds one focus, real or imaginary according as the directrix is real or imaginary. To each pair of parallel directrices corresponds one value of e^2.

2. In what follows Δ is supposed to have been made negative when its value is not zero. The positive value of R, say R', will then give the real values of p,

x', y': The equations of the four directrices having been found to be of the form

$$x\lambda + y\mu = p' \pm p'',$$

it follows that the equations of the two lines midway between the parallel directrices (the *axes* of the conic) are

$$x\lambda + y\mu = p', \tag{8}$$

where λ and μ have two values each. Substituting from (7) and reducing, (8) becomes

$$-2h\left(x + \frac{2g}{s - R}\right) + (R + d)\left(y + \frac{2f}{s - R}\right) = 0, \tag{9}$$

$$(R - d)\left(x + \frac{2g}{s - R}\right) - 2h\left(y + \frac{2f}{s - R}\right) = 0, \tag{10}$$

two forms of the same equation, of which one becomes indeterminate when $h = 0$. The positive value of R substituted either in (9) or in (10) gives the midway parallel to the real directrices (the *conjugate* axis), and the negative value of R gives likewise the midway parallel to the imaginary directrices, itself a real line (the *transverse* axis of the conic).

3. Now we might prove that the pair of real foci lie on the transverse axis by substituting their co-ordinates (7) in the equation of the axis, but the simplest way is as follows: If the transverse axis were taken as the axis of x and the conjugate as the axis of y, then we should have $\alpha_1 = 0$, $\alpha_2 = 90^\circ$, that is $\mu_1 = 0$, $\mu_2 = 1$; but this implies $h = 0$. Furthermore the two values of p belonging to each pair of parallel directrices would then differ only in sign. This implies $f = 0$, $g = 0$ in the equation of the conic as referred to the new axes. The co-ordinates of the two real foci would then become (7)

$$x_1' = \pm e_1^2 p_1'' \lambda_1 = \pm e^2 p'',$$
$$y_1' = \pm e_1^2 p_1'' \mu_1 = 0;$$

whence it is seen that the two real foci lie on the transverse axis at equal distances from the *centre* (intersection of the axes) of the conic. In the same way the two imaginary foci may be said to lie on the conjugate axis at equal imaginary distances from the centre, for their co-ordinates are

$$x_2' = 0,$$
$$y_2' = \pm e_2^2 p_2''.$$

From this it follows, as also from the fact that the new equation is of the form

$$a'x^2 + b'y^2 + c' = 0, \tag{11}$$

that a conic is symmetrical with respect to both of its axes.

From (7) we see that the co-ordinates of the centre are

$$x_0 = \frac{bg - fh}{h^2 - ab}, \tag{12}$$

$$y_0 = \frac{af - gh}{h^2 - ab}, \tag{13}$$

a result we might also obtain by finding where the axes intersect.

4. From (11) we see that a conic crosses each axis in two points at equal distances from the centre. Call this distance A. From the definition of a conic we know that these two points of crossing (*vertices*) divide the distance F from each focus (on this axis) to its directrix internally and externally in the ratio e. Then

$$\frac{A - F}{p'' - A} = \frac{A + F}{p'' + A} = e,$$

whence we get $A = ep''$ and $F = Ae$.

If a perpendicular be erected to an axis at a focus, then the distance on it to the curve (the *semi-parameter*) is by definition e times the distance from the focus to the corresponding directrix. Hence

$$\text{semi-parameter} = e\,(p'' - e^2 p'') = A\,(1 - e^2).$$

Collecting these metrical results we have

$$\begin{aligned}
\text{Directrix to centre} &= \sqrt{\frac{-4\,\Delta}{R\,(R - s)^2}},\\
\text{Semi-axis} = A &= \sqrt{\frac{-2\,\Delta}{(R - s)\,(h^2 - ab)}},\\
\text{Focus to centre} &= \sqrt{\frac{-R\,\Delta}{(h^2 - ab)^2}},\\
\text{Focus to directrix} &= \sqrt{\frac{-4\,\Delta}{R\,(R + s)^2}},\\
\text{Semi-parameter} &= \sqrt{\frac{-8\,\Delta}{(R + s)^3}}.
\end{aligned} \tag{14}$$

Each of these quantities has two values, a transverse and a conjugate value, arising from the positive and negative values of R respectively. Calling the two values of the semi-axis A_1 and A_2 respectively, we find $A_1^2 > A_2^2$ unless $R' = 0$, for these values may be written

$$A_1^2 = \frac{-8\,\Delta(s + R')}{(s^2 - R'^2)^2}, \quad A_2^2 = \frac{-8\,\Delta\,(s - R')}{(s^2 - R'^2)^2},$$

and $-\Delta$ is a positive quantity.

5. From (11) we get

$$\frac{x^2}{\dfrac{c'}{a'}} + \frac{y^2}{\dfrac{c'}{b'}} = 1,$$

where $-\dfrac{c'}{a'}$ and $-\dfrac{c'}{b'}$ are plainly the squares of the semi-axes of the conic (B). Hence the equation (B) when referred to the axes of the conic as axes of reference becomes

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1,$$

or

$$A_2^2x^2 + A_1^2y^2 - A_1^2A_2^2 = 0. \tag{15}$$

If we apply to (15) formulæ (7) we may get the same metrical results in terms of A_1 and A_2.

6. Let us, now, see what the equation of the conic (B) becomes when we take the transverse axis as the axis of x and a line at right angles through the left hand vertex as the axis of y; and let us for convenience form its equation with respect to that focus and directrix which lie on opposite sides of the axis of y. We have then $\lambda = 1$, $\mu = 0$, $y' = 0$, $x' = e_1p_1''(e_1 - 1)$ and $p = p_1''(1 - e_1)$. We do not and need not know in regard to the signs of x' and p whether they are positive or negative, but they are opposite by hypothesis. By substituting the above values in (A) we get

$$(1 - e_1^2)\,x^2 + y^2 = 2e_1p_1''\,(1 - e_1^2)\,x,$$

or

$$\frac{s - R'}{s + R'}\,x^2 + y^2 = 2\eta\,P_1x, \tag{16}$$

where η is the sign of $\dfrac{s - R'}{s + R'}$ and P_1 is the transverse semi-parameter.

Now it is clear that if the origin had been taken at the right hand transverse vertex, the resulting equation would have been the same as (16), except that the right hand member would have been negative; and in order to obtain the corresponding conjugate results, it is necessary only to substitute $-R'$ for $+R'$ in (16).

CLASSIFICATION OF CONICS.

In equations (15) and (16) as well as in the metrical results (14) we find involved only the quantities s, R', and Δ. The curves represented by (B) may therefore be classified according to the values of these three quantities.

When $R' < s$ numerically, then both values of e^2 are less than unity, one being positive, the other negative, and the curve is called an *ellipse.*

When $R' = s$, then the two values of e^2 are 1 and ∞, and the curve is called a *parabola.*

When $R' > s$ numerically, then both values of e^2 are greater than unity, and the curve is called an *hyperbola.*

Since $R' = \sqrt{s^2 + 4(h^2 - ab)}$, the above three conditions are equivalent to the following: —

$$h^2 - ab < 0 \text{ for the } \textit{ellipse},$$
$$h^2 - ab = 0 \text{ for the } \textit{parabola},$$
$$h^2 - ab > 0 \text{ for the } \textit{hyperbola};$$

except in the special case $a = b = h = 0$, when, although $h^2 - ab = 0$, $e^2 = \frac{0}{0}$. Equation (B) in this case degenerates to an equation of the first degree. Hence a straight line considered as a degenerate conic is to be regarded as belonging to either one of the three classes of conics.

Each of these three species of conics is subdivided according to the value of the product $R'sΔ$ as follows: —

CLASSIFICATION OF CURVES OF THE SECOND DEGREE.

		$h^2 - ab < 0$	$h^2 - ab = 0$	$h^2 - ab > 0$
$R'sΔ < 0$		Real, finite ellipse, axes unequal. $A_1 > A_2$.	True parabola.	Hyperbola with finite axes. $A_1^2 > -A_2^2$
	$R's = 0$ $Δ \neq 0$	$sΔ < 0$, Real circle. $sΔ > 0$, Imaginary circle. $A_1 = A_2$.	Conditions impossible.	Equilateral hyperbola. $A_1^2 = -A_2^2$
$R'sΔ = 0$	$R's \neq 0$ $Δ = 0$	Infinitesimal ellipse, axes unequal.	Two parallel lines. Real if $f^2 + g^2 - cs > 0$, Coincident, " $= 0$, Imaginary, " < 0.	Hyperbola with unequal infinitesimal axes. [Two intersecting lines.]
	$R's = 0$ $Δ = 0$	Infinitesimal circle.	Single right line.	Equilateral hyperbola with infinitesimal axes. [Two lines at right angles]
$R'sΔ > 0$		Imaginary ellipse. $A_1 < A_2$.	Conditions impossible.	Hyperbola with finite axes. $A_1^2 < -A_2^2$.

[TO BE CONTINUED.]